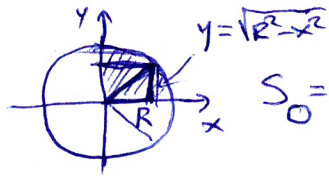


Examples on integration

① Area of a disc

- cartesian coordinates



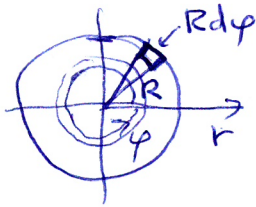
$$S_0 = 4S_{\Delta} = 4 \int_0^R \int_0^{\sqrt{R^2-x^2}} dy dx =$$

$$= 4 \left[\frac{1}{2} \left(x \sqrt{R^2-x^2} + R^2 \arctan \left(\frac{x}{\sqrt{R^2-x^2}} \right) \right) \right]_0^R =$$

$$= 2R^2 \lim_{x \rightarrow R} \arctan \left(\frac{x}{\sqrt{R^2-x^2}} \right) = 2R^2 \frac{\pi}{2} = \pi R^2$$

pretty ugly integral,
try to do the derivative
of the primitive
function :-)

- polar coordinates

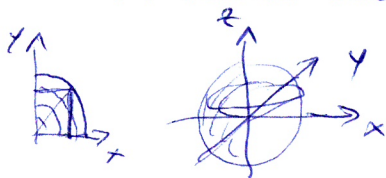


$$S_0 = \int_0^R \int_0^{2\pi} r dy dr = \int_0^R r dr \int_0^{2\pi} 1 dy = \int_0^R (2\pi r) dr =$$

$$= 2\pi \left[\frac{r^2}{2} \right]_0^R = \pi R^2$$

② Volume of a ball

- cartesian coordinates



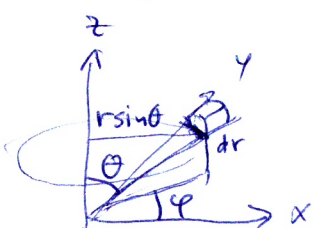
$$V = 8 \int_0^R \int_0^{\sqrt{R^2-x^2}} \int_0^{\sqrt{R^2-x^2-y^2}} dz dy dx = 8 \int_0^R \int_0^{\sqrt{R^2-x^2}} \frac{W^2}{W^2} dy dx =$$

$$= 8 \int_0^R dx \left(\frac{\pi W^2}{4} \right) = 2\pi \int_0^R (R^2-x^2) dx = 2\pi \left[R^2 x - \frac{x^3}{3} \right]_0^R =$$

$$= \frac{4}{3} \pi R^3$$

integral over y
is the same
as above
for R=W

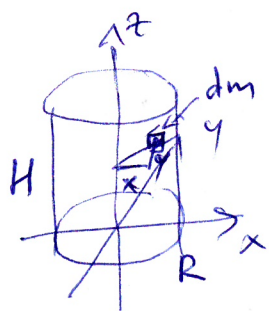
- spherical coordinates



$$V = \int_0^R \int_0^{\pi} \int_0^{2\pi} (r \sin \theta) d\phi = \int_0^R r^2 dr \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi$$

$$= \left[\frac{r^3}{3} \right]_0^R \cdot \left[-\cos \theta \right]_0^{\pi} \cdot \left[\phi \right]_0^{2\pi} = \frac{4\pi}{3} R^3$$

③ momentum of inertia of a cylinder about symmetry axis (z)



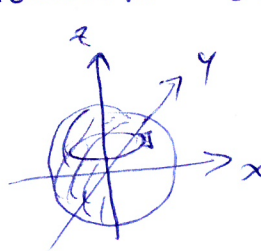
total mass $M = \rho \pi R^2 H$

$$I_z = \iiint_V \underbrace{(x^2 + y^2)}_{r^2} \underbrace{\rho}_{\text{constant}} dV = \rho \int_0^H dz \int_0^R dr \int_0^{2\pi} r^2 (r d\phi) =$$

$$= \rho \left[z \right]_0^H \cdot \left[\frac{r^4}{4} \right]_0^R \cdot \left[\phi \right]_0^{2\pi} =$$

$$= \rho H \frac{R^4}{4} \cdot 2\pi = \frac{1}{2} M R^2$$

④ momentum of inertia of a ball about z-axis



- in cylindrical coordinates

$$I_z = \iiint_V (x^2 + y^2) \rho dV = \text{twice half of a ball}$$

$$= 2 \int_0^R dz \int_0^{\sqrt{R^2 - z^2}} dr \int_0^{2\pi} r^2 \rho r d\phi = 4\pi \rho \int_0^R dz \int_0^{\sqrt{R^2 - z^2}} r^3 dr =$$

$$= 4\pi \rho \int_0^R \frac{(R^2 - z^2)^2}{4} dz = \pi \rho \left[R^4 z - 2R^2 \frac{z^3}{3} + \frac{z^5}{5} \right]_0^R =$$

$$= \pi \rho R^5 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{8}{15} \pi \rho R^5 = \frac{2}{5} M R^2$$

where $M = \frac{4}{3} \pi R^3 \rho$

- by symmetry trick and spherical coordinates

$$I_x = I_y = I_z = I$$

we sum $I_x + I_y + I_z = \iiint_V \underbrace{[(y^2 + z^2) + (x^2 + z^2) + (x^2 + y^2)]}_{2r^2} \rho dV =$

$$3I = 2\rho \int_0^R r^2 \cdot r dr \underbrace{\int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi}_{4\pi} = \frac{8\pi}{5} \rho R^5$$

or $I = \frac{8\pi}{15} \rho R^5 = \frac{2}{5} M R^2$